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Universes encircling 5-dimensional black holes

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Abstract

We clarify the status of two known solutions to the 5-dimensional vacuum Einstein field equations derived by Liu, Mashhoon & Wesson (LMW) and Fukui, Seahra & Wesson (FSW), respectively. Both 5-metrics explicitly embed 4-dimensional Friedman-Lemaître-Robertson-Walker cosmologies with a wide range of characteristics. We show that both metrics are also equivalent to 5-dimensional topological black hole (TBH) solutions, which is demonstrated by finding explicit coordinate transformations from the TBH to LMW and FSW line elements. We argue that the equivalence is a direct consequence of Birkhoff's theorem generalized to 5 dimensions. Finally, for a special choice of parameters we plot constant coordinate surfaces of the LMW patch in a Penrose-Carter diagram. This shows that the LMW coordinates are regular across the black and/or white hole horizons.

I. INTRODUCTION

Over the past few years, there has been a marked resurgence of interest in models with non-compact or large extra-dimensions. Three examples of such scenarios immediately come to mind — namely the braneworld models of Randall & Sundrum[1,2] (henceforth RS) and Arkani-Hamed, Dimopoulos & Dvali[3,4,5] (henceforth ADD), as well as the older Space-Time-Matter (STM) theory[6]. The RS model is motivated from certain ideas in string theory, which suggest that the particles and fields of the standard model are naturally confined to a lower-dimensional hypersurface living in a non-compact, higher-dimensional bulk manifold. The driving goal behind the ADD picture is to explain the discrepancy in scale between the observed strength of the gravitational interaction and the other fundamental forces. This is accomplished by noting that in generic higher-dimensional models with compact extra dimensions, the bulk Newton's constant is related to the effective 4-dimensional constant by factors depending on the size and number of the extra dimensions. Finally, STM or induced matter theory proposes that our universe is an embedded 4-surface in a vacuum 5-manifold. In this picture, what we perceive to be the source in the 4-dimensional Einstein field equations is really just an artifact of the embedding; or in other words, conventional matter is induced from higher-dimensional geometry.

Regardless of the motivation, if extra dimensions are to be taken seriously then it is useful to have as many solutions of the higher-dimensional Einstein equations at our disposal as possible. These metrics serve as both arenas in which to test the feasibility of extra dimensions, as well as guides as to where 4-dimensional general relativity may break down. This simplest type of higher-dimensional field equations that one might consider is the 5-dimensional vacuum field equations $\hat{R}_{AB} = 0$. (In this paper, uppercase Latin indices run $0 \ldots 4$ while lowercase Greek indices run $0 \ldots 3$, and 5-dimensional curvature tensors are distinguished from the 4-dimensional counterparts by hats. Also, commas in subscripts indicate partial differentiation.) This condition is most relevant to the STM scenario, but can also be applied to the RS or ADD pictures. The are a fair number of known solutions that embed 4-manifolds of cosmological or spherically-symmetric character; one can consult the book by Wesson[6] for an accounting of these metrics.

However, when searching for new solutions to vacuum field equations, one must keep in mind a known peril from 4-dimensional work; i.e., any new solution could be a previously discovered metric written down in terms of strange coordinates. Our purpose in this paper is to demonstrate that two 5-dimensional vacuum solutions in the literature are actually isometric to a generalized 5-dimensional Schwarzschild manifold. Both of these solutions have been previously analyzed in the context of 4-dimensional cosmology because they both embed submanifolds with line elements matching that of standard Friedman-Lemaître-Robertson-Walker (FLRW) models with flat, spherical, or hyperbolic spatial sections. In Section II A, we discuss the first of these 5-metrics, which was originally written down by Liu & Mashhoon[7] and later rediscovered in a different form by Liu & Wesson[8]. We will see that this metric naturally embeds FLRW models with fairly general, but not unrestricted, scale factor behaviour. Several different authors have considered this metric in a number of different contexts[9,10,11,12], including the RS braneworld scenario. The second 5-metric — which was discovered by Fukui, Seahra & Wesson[13] and is the subject of Section II B — also embeds FLRW models with all types of spatial curvature, but the scale factor is much more constrained. We will pay special attention to the characteristics of the embedded cosmologies in each solution, as well as the coordinate invariant geometric properties of the associated

bulk manifolds.

The latter discussion will reveal that not only do the Liu-Mashhoon-Wesson (LMW) and Fukui-Seahra-Wesson (FSW) metrics have a lot in common with one another, they also exhibit many properties similar to that of the topological black hole (TBH) solution of the 5-dimensional vacuum field equations, which we introduce in Section III. This prompts us to suspect that the LMW and FSW solutions are actually isometric to topological black hole manifolds. We confirm this explicitly by finding transformations from standard black hole to LMW and FSW coordinates in Sections IV A and IV B respectively. We argue that the equivalence of the three metrics is actually a consequence of a higher-dimensional version of Birkhoff's theorem in Section IV C. In Section V, we discuss which portion of the extended 5-dimensional Kruskal manifold is covered by the LMW coordinate patch and obtain Penrose-Carter embedding diagrams for a particular case. Section VI summarizes and discusses our results.

II. TWO 5-METRICS WITH FLRW SUBMANIFOLDS

In this section, we introduce two 5-metrics that embed 4-dimensional FLRW models. Both of these are solutions of the 5-dimensional vacuum field equations, and are hence suitable manifolds for STM theory. Our goals are to illustrate what subset of all possible FLRW models can be realized as hypersurfaces contained within these manifolds, and to find out about any 5-dimensional curvature singularities or geometric features that may be present.

A. The Liu-Mashhoon-Wesson metric

Consider a 5-dimensional manifold (M_{LMW}, g_{AB}). We define the LMW metric *ansatz* as:

$$ds^2_{\text{LMW}} = \frac{a^2_{,t}(t,\ell)}{\mu^2(t)} dt^2 - a^2(t,\ell)\, d\sigma^2_{(k,3)} - d\ell^2. \tag{1}$$

Here, $a(t,\ell)$ and $\mu(t)$ are undetermined functions, and $d\sigma^2_{(k,3)}$ is the line element on maximally symmetric 3-spaces $\mathbb{S}^{(k)}_3$ with curvature index $k = +1, 0, -1$:

$$d\sigma^2_{(k,3)} = d\psi^2 + S^2_k(\psi)(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2}$$

where

$$S_k(\psi) \equiv \begin{cases} \sin\psi, & k = +1, \\ \psi, & k = 0, \\ \sinh\psi, & k = -1, \end{cases} \tag{3}$$

It is immediately obvious that the ℓ = constant hypersurfaces Σ_ℓ associated with (1) have the structure of FLRW models: $\mathbb{R} \times \mathbb{S}^{(k)}_3$. We should note that the original papers (refs. 7 and 8) did not really begin with a metric *ansatz* like (1); rather, the g_{tt} component of the metric was initially taken to be some general function of t and ℓ. But one rapidly closes in on the above line element by direct integration of one component of the vacuum field equations $\hat{R}_{AB} = 0$; namely, $\hat{R}_{t\ell} = 0$. The other components are satisfied if

$$a^2(t,\ell) = [\mu^2(t) + k]\ell^2 + 2\nu(t)\ell + \frac{\nu^2(t) + \mathcal{K}}{\mu^2(t) + k}, \tag{4}$$

where $\mathcal{K}$ is an integration constant. As far as the field equations are concerned, $\mu(t)$ and $\nu(t)$ are *completely arbitrary* functions of time. However, we should constrain them by appending the condition

$$a(t,\ell) \in \mathbb{R}^+ \quad \Rightarrow \quad a^2(t,\ell) > 0 \tag{5}$$

to the system. This restriction ensures that the metric signature is $(+----)$ and t is the only timelike coordinate. Now, if a is taken to be real, then it follows that ν must be real as well. Regarding (4) as a quadratic equation in ν, we find that there are real solutions only if the quadratic discriminant is non-negative. This condition translates into

$$\mathcal{K} \leq a^2(t,\ell)[\mu^2(t) + k]. \tag{6}$$

If $\mathcal{K}$ is positive this inequality implies that we must choose $\mu(t)$ such that $\mu^2 + k > 0$. This relation will be important shortly.

The reason that this solution is of interest is that the induced metric on $\ell =$ constant hypersurfaces is isometric to the standard FLRW line element. To see this explicitly, consider the line element on the $\ell = \ell_0$ 4-surface:

$$ds^2_{(\Sigma_\ell)} = \frac{a^2_{,t}(t,\ell_0)}{\mu^2(t)} dt^2 - a^2(t,\ell_0)\, d\sigma^2_{(k,3)}. \tag{7}$$

Let us perform the 4-dimensional coordinate transformation

$$\Theta(t) = \int_t \frac{a_{,u}(u,\ell_0)}{\mu(u)} du \quad \Rightarrow \quad \mu(t(\Theta)) = \mathcal{A}'(\Theta), \tag{8}$$

where

$$\mathcal{A}(\Theta) = a(t(\Theta),\ell_0), \tag{9}$$

and we use a prime to denote the derivative of functions of a single argument. This puts the induced metric in the FLRW form

$$ds^2_{(\Sigma_\ell)} = d\Theta^2 - \mathcal{A}^2(\Theta)\, d\sigma^2_{(k,3)}, \tag{10}$$

where Θ is the cosmic time and $\mathcal{A}(\Theta)$ is the scale factor.

So, the geometry of each of the Σ_ℓ hypersurfaces is indeed of the FLRW-type. But what kind of cosmologies can be thus embedded? Well, if we rewrite the inequality (6) in terms of $\mathcal{A}$ and $\mathcal{A}'$ we obtain

$$\mathcal{K} \leq \mathcal{A}^2(\mathcal{A}'^2 + k). \tag{11}$$

Since $\mathcal{A}$ is to be interpreted as the scale factor of some cosmological model, it satisfies the Friedman equation:

$$\mathcal{A}'^2 - \tfrac{1}{3}\kappa_4^2 \rho \mathcal{A}^2 = -k. \tag{12}$$

Here, ρ is the total density of the matter-energy in the cosmological model characterized by $\mathcal{A}(\Theta)$ and $\kappa_4^2 = 8\pi G$ is the usual coupling constant in the 4-dimensional Einstein equations. This implies a relation between the density of the embedded cosmologies and the choice of μ:

$$\mu^2 + k = \tfrac{1}{3}\kappa_4^2 \rho \mathcal{A}^2. \tag{13}$$

This into the inequality (11) yields

$$\mathcal{K} \leq \tfrac{1}{3}\kappa_4^2 \rho \mathcal{A}^4. \tag{14}$$

Therefore, we can successfully embed a given FLRW model on a Σ_ℓ 4-surface in the LMW solution if the total density of the model's cosmological fluid and scale factor satisfy (14) for all Θ. An obvious corollary of this is that we can embed any FLRW model with $\rho > 0$ if $\mathcal{K} < 0$.

There is one other point about the intrinsic geometry of the Σ_ℓ hypersurfaces that needs to be made. Notice that our 4-dimension coordinate transformation (8) has

$$\frac{d\Theta}{dt} = \frac{a_{,t}}{\mu}, \tag{15}$$

which means that the associated Jacobian vanishes whenever $a_{,t} = 0$. Therefore, the transformation is really only valid in between the turning points of a. Also notice that the original 4-metric (7) is badly behaved when $a_{,t} = 0$, but the transformed one (10) is not when $\mathcal{A}' = 0$. We can confirm via direct calculation that the Ricci scalar for (7) is

$${}^{(4)}R = -\frac{6\mu}{a}\frac{d\mu}{dt}\left(\frac{\partial a}{\partial t}\right)^{-1} - \frac{6}{a^2}(\mu^2 + k). \tag{16}$$

We see that ${}^{(4)}R$ diverges when $a_{,t} = 0$, provided that $\mu\mu_{,t}/a \neq 0$. Therefore, there can be genuine curvature singularities in the intrinsic 4-geometry at the turning points of a. These features are hidden in the altered line element (10) because the coordinate transformation (8) is not valid in the immediate vicinity of any singularities, hence the Θ-patch cannot cover those regions (if they exist). We mention that this 4-dimensional singularity in the LMW metric has been recently investigated by Xu, Liu and Wang[14], who have interpreted it as a 4-dimensional event horizon.

Now, let us turn our attention to some of the 5-dimensional geometric properties of M_{LMW}. We can test for curvature singularities in this 5-manifold by calculating the Kretschmann scalar:

$$\mathfrak{K}_{\text{LMW}} \equiv \hat{R}^{ABCD}\hat{R}_{ABCD} = \frac{72\mathcal{K}^2}{a^8(t,\ell)}. \tag{17}$$

We see there is a singularity in the 5-geometry along the hypersurface $a(t,\ell) = 0$. (Of course, whether or not $a(t,\ell) = 0$ for any $(t,\ell) \in \mathbb{R}^2$ depends on the choice of μ and ν.) This singularity is essentially a line-like object because the radius a of the 3-dimensional $\mathbb{S}_3^{(k)}$ subspace vanishes there. Other tools for probing the 5-geometry are Killing vector fields on M_{LMW}. Now, there are by definition 6 Killing vectors associated with symmetry operations on $\mathbb{S}_3^{(k)}$, but there is also at least one Killing vector that is orthogonal to that submanifold. This vector field is given by

$$\xi_A^{\text{LMW}} dx^A = \frac{a_{,t}}{\mu}\sqrt{h(a) + \mu^2(t)}\, dt + \sqrt{a_{,\ell}^2 - h(a)}\, d\ell. \tag{18}$$

Here, we have defined

$$h(x) \equiv k - \frac{\mathcal{K}}{x^2}. \tag{19}$$

Using the explicit form of $a(t,\ell)$ from equation (4), we can verify that ξ satisfies Killing's equation

$$\nabla_B \xi_A^{\text{LMW}} + \nabla_A \xi_B^{\text{LMW}} = 0, \tag{20}$$

via computer. Also using (4), we can calculate the norm of ξ^{LMW}, which is given by

$$\xi^{\text{LMW}} \cdot \xi^{\text{LMW}} = h(a). \tag{21}$$

This vanishes at $ka^2 = \mathcal{K}$. So, if $k\mathcal{K} > 0$ the 5-manifold contains a Killing horizon. If the horizon exists then ξ^{LMW} will be timelike for $|a| > \sqrt{|\mathcal{K}|}$ and spacelike for $|a| < \sqrt{|\mathcal{K}|}$.

To summarize, we have seen that FLRW models satisfying (14) can be embedded on a Σ_ℓ 4-surface within the LMW metric, but that there are 4-dimensional curvature singularities wherever $a_{,t} = 0$. The LMW 5-geometry also possesses a line-like singularity where $a(t, \ell) = 0$, as well as a Killing horizon across which the norm of ξ^{LMW} changes sign.

B. The Fukui-Seahra-Wesson metric

For the time being, let us set aside the LMW metric and concentrate on the FSW solution. On a certain 5-manifold (M_{FSW}, g_{AB}), this is given by the line element

$$ds^2_{\text{FSW}} = d\tau^2 - b^2(\tau, w)\, d\sigma^2_{(k,3)} - \frac{b^2_{,w}(\tau, w)}{\zeta^2(w)} dw^2, \tag{22a}$$

$$b^2(\tau, w) = [\zeta^2(w) - k]\tau^2 + 2\chi(w)\tau + \frac{\chi^2(w) - \mathcal{K}}{\zeta^2(w) - k}. \tag{22b}$$

This metric (22a) is a solution of the 5-dimensional vacuum field equations $\hat{R}_{AB} = 0$ with $\zeta(w)$ and $\chi(w)$ as arbitrary functions. Just as before, we call equation (22a) the FSW metric *ansatz*, even though it was not the technical starting point of the original paper[13]. We have written (22) in a form somewhat different from that of ref. 13; to make contact with their notation we need to make the correspondences

$$[F(w)]_{\text{FSW}} \equiv k - \zeta^2(w), \tag{23a}$$

$$[h(w)]_{\text{FSW}} \equiv [\chi^2(w) + \mathcal{K}]/[\zeta^2(w) - k], \tag{23b}$$

$$[g(w)]_{\text{FSW}} \equiv 2\chi(w), \tag{23c}$$

$$[\mathcal{K}]_{\text{FSW}} \equiv -4\mathcal{K}, \tag{23d}$$

where $[\cdots]_{\text{FSW}}$ indicates a quantity from the original FSW work. A cursory comparison between the LMW and FSW vacuum solutions reveals that both metrics have a similar structure, which prompts us to wonder about any sort of fundamental connection between them. We defer this issue to the next section, and presently concern ourselves with the properties of the FSW solution in its own right.

Just as for the LMW metric, we can identify hypersurfaces in the FSW solution with FLRW models. Specifically, the induced metric on $w = w_0$ hypersurfaces Σ_w is

$$ds^2_{(\Sigma_w)} = d\tau^2 - b^2(\tau, w_0)\, d\sigma^2_{(k,3)}. \tag{24}$$

We see that for the universes on Σ_w, τ is the cosmic time and $b(\tau, w_0)$ is the scale factor. It is useful to perform the following linear transformation on τ:

$$\tau(\Theta) = \Theta - \frac{\chi_0}{\zeta_0^2 - k}, \tag{25}$$

	$\zeta_0^2 - k > 0$	$\zeta_0^2 - k < 0$
$\mathcal{K} > 0$	big bang	big bang and big crunch
$\mathcal{K} = 0$	big bang	$\mathcal{B} \in \mathbb{C}$ for all $\Theta \in \mathbb{R}$
$\mathcal{K} < 0$	no big bang/crunch	$\mathcal{B} \in \mathbb{C}$ for all $\Theta \in \mathbb{R}$

TABLE I: Characteristics of the 4-dimensional cosmologies embedded on the Σ_w hypersurfaces in the FSW metric

where we have defined $\zeta_0 \equiv \zeta(w_0)$ and $\chi_0 \equiv \chi(w_0)$. This puts the induced metric into the form

$$ds^2_{(\Sigma_w)} = d\Theta^2 - \mathcal{B}^2(\Theta)\, d\sigma^2_{(k,3)}, \tag{26a}$$

$$\mathcal{B}(\Theta) = \sqrt{\frac{(\zeta_0^2 - k)^2\Theta^2 - \mathcal{K}}{\zeta_0^2 - k}}. \tag{26b}$$

Unlike the LMW case, the cosmology on the Σ_w hypersurfaces has restrictive properties. If $\zeta_0^2 - k > 0$, the scale factor $\mathcal{B}(\Theta)$ has the shape of one arm of a hyperbola with a semi-major axis of length $\sqrt{-\mathcal{K}/(\zeta_0^2 - k)}$. Note that this length may be complex depending on the values of ζ_0, k and $\mathcal{K}$. That is, the scale factor may not be defined for all $\Theta \in \mathbb{R}$. When this is the case, the embedded cosmologies involve a big bang and/or a big crunch. Conversely, it is not hard to see if $\zeta_0^2 - k < 0$ and $\mathcal{K} > 0$ then the cosmology is re-collapsing; i.e., there is a big bang and a big crunch. However, if $\zeta_0^2 - k < 0$ and $\mathcal{K} \leq 0$, then there is no Θ interval where the scale factor is real. We have summarized the basic properties of the embedded cosmologies in Table I. Finally, we note that if $\zeta_0^2 - k > 0$ then

$$\lim_{\Theta \to \infty} \mathcal{B}(\Theta) = (\zeta_0^2 - k)^{1/2}\Theta. \tag{27}$$

Hence, the late time behaviour of such models approaches that of the empty Milne universe.

Lake[15] has calculated the Kretschmann scalar for vacuum 5-metrics of the FSW type. When his formula is applied to (22), we obtain:

$$\mathfrak{K}_{\text{FSW}} \equiv \hat{R}^{ABCD}\hat{R}_{ABCD} = \frac{72\mathcal{K}^2}{b^8(\tau, w)}. \tag{28}$$

As for the LMW manifold, this implies the existence of a line-like singularity in the 5-geometry at $b(\tau, w) = 0$. We also find that there is a Killing vector on M_{FSW}, which is given by

$$\xi_A^{\text{FSW}} dx^A = \sqrt{b_{,\tau} + h(b)}\, d\tau + \frac{b_{,w}}{\zeta}\sqrt{\zeta^2 - h(b)}\, dw \tag{29a}$$

$$0 = \nabla_A \xi_B^{\text{FSW}} + \nabla_B \xi_A^{\text{FSW}}. \tag{29b}$$

The norm of this Killing vector is relatively easily found by computer:

$$\xi^{\text{FSW}} \cdot \xi^{\text{FSW}} = h(b). \tag{30}$$

Hence, there is a Killing horizon in M_{FSW} where $h(b) = 0$. Obviously, the ξ^{FSW} Killing vector changes from timelike to spacelike — or *vice versa* — as the horizon is traversed.

In summary, we have seen how FLRW models with scale factors of the type (22b) are embedded in the FSW solution. We found that there is a line-like curvature singularity in M_{FSW} at $b(\tau, w) = 0$ and the bulk manifold has a Killing horizon where the magnitude of ξ^{FSW} vanishes.

III. CONNECTION TO THE 5-DIMENSIONAL TOPOLOGICAL BLACK HOLE MANIFOLD

When comparing equations (17) and (28), or (21) and (30), it is hard not to believe that there is some sort of fundamental connection between the LMW and FSW metrics. We see that

$$\mathfrak{K}_{\text{LMW}} = \mathfrak{K}_{\text{FSW}}, \quad \xi^{\text{LMW}} \cdot \xi^{\text{LMW}} = \xi^{\text{FSW}} \cdot \xi^{\text{FSW}}, \tag{31}$$

if we identify $a(t,\ell) = b(\tau, w)$. Also, we notice that the LMW solution can be converted into the FSW metric by the following set of transformations/Wick rotations[16]:

$$\begin{aligned} \psi &\to i\psi, & t &\to w, \\ \ell &\to \tau, & k &\to -k, \\ \mathcal{K} &\to -\mathcal{K}, & ds_{\text{LMW}} &\to i\, ds_{\text{FSW}}. \end{aligned} \tag{32}$$

These facts lead us to the strong suspicion that the LMW and FSW metrics actually describe the same 5-manifold.

But which 5-manifold might this be? We established in the previous section that both the LMW and FSW metrics involve a 5-dimensional line-like curvature singularity and Killing horizon if $k\mathcal{K} > 0$. This reminds us of another familiar manifold: that of a black hole. Consider the metric of a "topological" black hole (TBH) on a 5-manifold (M_{TBH}, g_{AB}):

$$ds^2_{\text{TBH}} = h(R)\, dT^2 - h^{-1}(R)\, dR^2 - R^2\, d\sigma^2_{(k,3)}. \tag{33}$$

The adjective "topological" comes from the fact that the manifold has the structure $\mathbb{R}^2 \times \mathbb{S}_3^{(k)}$, as opposed to the familiar $\mathbb{R}^2 \times S_3$ structure commonly associated with spherical symmetry in 5-dimensions. That is, the surfaces $T =$ constant and $R =$ constant are not necessarily 3-spheres for the topological black hole; it is possible that they have flat or hyperbolic geometry. One can confirm by direct calculation that (33) is a solution of $\hat{R}_{AB} = 0$ for any value of k, and that the constant $\mathcal{K}$ that appears in $h(R)$ is related to the mass of the central object. The Kretschmann scalar on M_{TBH} is

$$\mathfrak{K}_{\text{TBH}} = \hat{R}^{ABCD} \hat{R}_{ABCD} = \frac{72\mathcal{K}^2}{R^8}, \tag{34}$$

implying a line-like curvature singularity at $R = 0$. There is an obvious Killing vector in this manifold, given by

$$\xi^{\text{TBH}}_A dx^A = h(R)\, dT. \tag{35}$$

The norm of this vector is trivially

$$\xi^{\text{TBH}} \cdot \xi^{\text{TBH}} = h(R). \tag{36}$$

There is therefore a Killing horizon in this space located at $kR^2 = \mathcal{K}$.

Now, equations (34) and (36) closely match their counterparts for the LMW and FSW metrics, which inspires the hypothesis that not only are the LMW and FSW isometric to one another, they are also isometric to the metric describing topological black holes. However, while these coincidences provide fairly compelling circumstantial evidence that the LMW, FSW, and TBH metrics are equivalent, we do not have conclusive proof — that will come in the next section.

IV. COORDINATE TRANSFORMATIONS

In this section, our goal is to prove the conjecture that the LMW, FSW, and TBH solutions and the 5-dimensional vacuum field equations are isometric to one another. We will do so by finding two explicit coordinate transformations that convert the TBH metric to the LMW and FSW metrics respectively. This is sufficient to prove the equality of all three solutions, since it implies that one can transform from the LMW to the FSW metric — or *vice versa* — via a two-stage procedure.

A. Transformation from Schwarzschild to Liu-Mashhoon-Wesson coordinates

We first search for a coordinate transformation that takes the TBH line element (33) to the LMW line element (1). We take this transformation to be

$$R = \mathcal{R}(t,\ell), \quad T = \mathcal{T}(t,\ell). \tag{37}$$

Notice that we have *not* assumed $R = a(t,\ell)$ — as may have been expected from the discussion of the previous section — in order to stress that we are starting with a general coordinate transformation. We will soon see that by demanding that this transformation forces the TBH metric into the form of the LMW metric *ansatz*, we can recover $R = a(t,\ell)$ with $a(t,\ell)$ given explicitly by (4). In other words, the coordinate transformation specified in this section will fix the functional form of $a(t,\ell)$ in a manner independent of the direct attack on the vacuum field equations found in refs. 7 and 8.

When (37) is substituted into (33), we get

$$\begin{aligned} ds^2_{\text{TBH}} = {} & \left[h(\mathcal{R})\mathcal{T}^2_{,t} - \frac{\mathcal{R}^2_{,t}}{h(\mathcal{R})}\right] dt^2 + 2\left[h(\mathcal{R})\mathcal{T}_{,t}\mathcal{T}_{,\ell} - \frac{\mathcal{R}_{,t}\mathcal{R}_{,\ell}}{h(\mathcal{R})}\right] dt\, d\ell \, + \\ & \left[h(\mathcal{R})\mathcal{T}^2_{,\ell} - \frac{\mathcal{R}^2_{,\ell}}{h(\mathcal{R})}\right] d\ell^2 - \mathcal{R}^2(t,\ell)\, d\sigma^2_{(k,3)}. \end{aligned} \tag{38}$$

For this to match equation (1) with $\mathcal{R}(t,\ell)$ instead of $a(t,\ell)$ we must have

$$\frac{\mathcal{R}^2_{,t}}{\mu^2(t)} = h(\mathcal{R})\mathcal{T}^2_{,t} - \frac{\mathcal{R}^2_{,t}}{h(\mathcal{R})}, \tag{39a}$$

$$0 = h(\mathcal{R})\mathcal{T}_{,t}\mathcal{T}_{,\ell} - \frac{\mathcal{R}_{,t}\mathcal{R}_{,\ell}}{h(\mathcal{R})}, \tag{39b}$$

$$-1 = h(\mathcal{R})\mathcal{T}^2_{,\ell} - \frac{\mathcal{R}^2_{,\ell}}{h(\mathcal{R})}, \tag{39c}$$

with $\mu(t)$ arbitrary. Under these conditions, we find

$$ds^2_{\text{TBH}} = \frac{\mathcal{R}^2_{,t}(t,\ell)}{\mu^2(t)} dt^2 - \mathcal{R}^2(t,\ell)\, d\sigma^2_{(k,3)} - dy^2, \tag{40}$$

which is obviously the same as the LMW metric *ansatz* (1). However, the precise functional form of $\mathcal{R}(t,\ell)$ has yet to be specified.

To solve for $\mathcal{R}(t,\ell)$, we note equations (39a) and (39c) can be rearranged to give

$$\mathcal{T}_{,t} = \epsilon_t \frac{\mathcal{R}_{,t}}{h(\mathcal{R})}\sqrt{1+\frac{h(\mathcal{R})}{\mu^2(t)}}, \tag{41a}$$

$$\mathcal{T}_{,\ell} = \epsilon_\ell \frac{1}{h(\mathcal{R})}\sqrt{\mathcal{R}_{,\ell}^2 - h(\mathcal{R})}, \tag{41b}$$

where $\epsilon_t = \pm 1$ and $\epsilon_\ell = \pm 1$. Using these in (39b) yields

$$\mathcal{R}_{,\ell} = \pm\sqrt{h(\mathcal{R}) + \mu^2(t)}. \tag{42}$$

Our task it to solve the system of PDEs formed by equations (41) and (42) for $\mathcal{T}(t,\ell)$ and $\mathcal{R}(t,\ell)$. Once we have accomplished this, the coordinate transformation from (1) to (33) is found.

Using the definition of $h(\mathcal{R})$, we can expand equation (42) to get

$$\pm 1 = \frac{\mathcal{R}}{\sqrt{(\mu^2+k)\mathcal{R}^2 - \mathcal{K}}}\frac{\partial \mathcal{R}}{\partial \ell}. \tag{43}$$

Integrating both sides with respect to ℓ yields

$$\sqrt{(\mu^2+k)\mathcal{R}^2 - \mathcal{K}} = (\mu^2+k)(\pm\ell + \gamma), \tag{44}$$

where $\gamma = \gamma(t)$ is an arbitrary function of time. Solving for $\mathcal{R}$ gives

$$R^2 = \mathcal{R}^2(t,\ell) = [\mu^2(t)+k]\ell^2 + 2\nu(t)\ell + \frac{\nu^2(t)+\mathcal{K}}{\mu^2(t)+k}, \tag{45}$$

where we have defined

$$\nu(t) = \pm\gamma(t)[\mu^2(t)+k], \tag{46}$$

which can be thought of as just an another arbitrary function of time. We have hence see that the functional form of $\mathcal{R}(t,\ell)$ matches exactly the functional form of $a(t,\ell)$ in equation (4). This is despite the fact that the two expressions were derived by different means: (45) from conditions placed on a coordinate transformation, and (4) from the direct solution of the 5-dimensional vacuum field equations.

When our solution for $\mathcal{R}(t,\ell)$ is put into equations (41), we obtain a pair of PDEs that expresses the gradient of $\mathcal{T}$ in the (t,ℓ)-plane as known functions of the coordinates. This is analogous to a problem where one is presented with the components of a 2-dimensional force and is asked to find the associated potential. The condition for integrablity of the system is that the curl of the force vanishes, which in our case reads

$$0 \stackrel{?}{=} \epsilon_t \frac{\partial}{\partial \ell}\left(\frac{\mathcal{R}_{,t}}{h(\mathcal{R})}\sqrt{1+\frac{h(\mathcal{R})}{\mu^2(t)}}\right) - \epsilon_\ell \frac{\partial}{\partial t}\left(\frac{1}{h(\mathcal{R})}\sqrt{\mathcal{R}_{,\ell}^2 - h(\mathcal{R})}\right). \tag{47}$$

We have confirmed via computer that this condition holds when $\mathcal{R}(t,\ell)$ is given by equation (45), provide we choose $\epsilon_t = \epsilon_\ell = \pm 1$. Without loss of generality, we can set $\epsilon_t = \epsilon_\ell = 1$. Hence, equations (41) are indeed solvable for $\mathcal{T}(t,\ell)$ and a *coordinate transformation from (33) to (1) exists.*

The only thing left is the tedious task of determining the explicit form of $\mathcal{T}(t,\ell)$. We spare the reader the details and just quote the solution, which can be checked by explicit substitution into (41). For $k = \pm 1$, we get

$$\mathcal{T}(t,\ell) = \frac{1}{k}\int_t \left\{ \frac{1}{\mu(u)}\frac{d}{du}\nu(u) - \left[\frac{\nu(u)}{\mu^2(u)+k}\right]\frac{d}{du}\mu(u)\right\} du + \frac{1}{k}\left(\mu(t)\ell - \frac{\mathcal{K}}{2\sqrt{k\mathcal{K}}}\ln\frac{1+\mathcal{X}(t,\ell)}{1-\mathcal{X}(t,\ell)}\right), \tag{48a}$$

$$\mathcal{X}(t,\ell) \equiv \frac{k}{\sqrt{k\mathcal{K}}}\frac{[\mu^2(t)+k]\ell + \nu(t)}{\mu(t)}. \tag{48b}$$

For $k = 0$, we obtain

$$\mathcal{T}(t,\ell) = \frac{1}{\mathcal{K}}\int_t \left\{ \frac{\nu^2(u)}{\mu^3(u)}\frac{d}{du}\nu(u) - \frac{\nu(u)[\nu^2(u)+\mathcal{K}]}{\mu^4(u)}\frac{d}{du}\mu(u)\right\} du + \frac{1}{\mathcal{K}}\left\{\frac{1}{3}\mu^3(t)\ell^3 + \mu(t)\nu(t)\ell^2 + \left[\frac{\nu^2(t)+\mathcal{K}}{\mu(t)}\right]\ell\right\}. \tag{49}$$

Recall that in these expression, μ and ν can be regarded as free functions. Taken with (45), these equations give the transformation from TBH to LMW coordinates explicitly.

Before moving on, there is one special case that we want to highlight. This is defined by $k\mathcal{K} < 0$, which implies that there is no Killing horizon in the bulk for real values of R and we have a naked singularity. If we have a spherical 3-geometry, then this is the case of a negative mass black hole. We have that $\sqrt{k\mathcal{K}} = i\sqrt{-k\mathcal{K}}$, which allows us to rewrite equation (48) as

$$\mathcal{T}(t,\ell) = \frac{1}{k}\left\{\mu(t)\ell + \frac{\mathcal{K}}{\sqrt{-k\mathcal{K}}}\arctan\left(\frac{k}{\sqrt{-k\mathcal{K}}}\frac{[\mu^2(t)+k]\ell+\nu(t)}{\mu(t)}\right)\right\} + \frac{1}{k}\int_t \left\{ \frac{1}{\mu(u)}\frac{d}{du}\nu(u) - \left[\frac{\nu(u)}{\mu^2(u)+k}\right]\frac{d}{du}\mu(u)\right\} du. \tag{50}$$

In obtaining this, we have made use of the identity

$$\arctan z = \frac{1}{2i}\ln\frac{1+iz}{1-iz}, \quad z \in \mathbb{C}. \tag{51}$$

To summarize this section, we have successfully found a coordinate transformation between the TBH to LMW coordinates. This establishes that those two solutions are indeed isometric, and are hence equivalent.

B. Transformation from Schwarzschild to Fukui-Seahra-Wesson coordinates

We now turn our attention to finding a transformation between the TBH and FSW line elements. The procedure is very similar to the one presented in the previous section. We begin by applying the following general coordinate transformation to the TBH solution (33):

$$T = \mathsf{T}(\tau, w), \quad R = \mathsf{R}(\tau, w). \tag{52}$$

Again, instead of identifying $\mathsf{R}(\tau, w) = b(\tau, w)$ as given by (22b), we regard it as a function to be solved for. To match the metric resulting from this transformation with (22a) we demand

$$+1 = h(\mathsf{R})\mathsf{T}_{,\tau}^2 - \frac{\mathsf{R}_{,t}^2}{h(\mathsf{R})}, \tag{53a}$$

$$0 = h(\mathsf{R})\mathsf{T}_{,\tau}\mathsf{T}_{,w} - \frac{\mathsf{R}_{,\tau}\mathsf{R}_{,w}}{h(\mathsf{R})}, \tag{53b}$$

$$-\frac{\mathsf{R}_{,w}^2}{\zeta^2(w)} = h(\mathsf{R})\mathsf{T}_{,w}^2 - \frac{\mathsf{R}_{,w}^2}{h(\mathsf{R})}. \tag{53c}$$

Here, $\zeta(w)$ is an arbitrary function. Compare this to the previous system of PDEs (39). We have essentially swapped and changed the signs of the lefthand sides of (39a) and (39c), as well as replaced $\mathcal{R}_{,t}$ with $\mathsf{R}_{,w}$ and $\mu(t)$ with $\zeta(w)$. This constitutes a sort of identity exchange $t \to w$ and $\ell \to \tau$. The explicit form of the TBH metric after this transformation is applied is

$$ds^2_{\text{TBH}} = d\tau^2 - \mathsf{R}^2(\tau, w)\, d\sigma^2_{(k,3)} - \left[\frac{\mathsf{R}_{,w}(t, w)}{\zeta(w)}\right]^2 dw^2. \tag{54}$$

This matches the FSW metric *ansatz* (22a), but the functional form of $\mathsf{R}(\tau, w)$ is yet to be determined by the coordinate transformation (53).

Let us now determine it by repeating the manipulations of the last section. We find that R satisfies the PDE

$$\mathsf{R}_{,\tau} = \pm\sqrt{\zeta^2(w) - h(\mathsf{R})}, \tag{55}$$

which is solved by

$$\mathsf{R}^2(\tau, w) = [\zeta^2(w) - k]\tau^2 + 2\chi(w)\tau + \frac{\chi^2(w) - \mathcal{K}}{\zeta^2(w) - k}. \tag{56}$$

Here, χ is an arbitrary function. In a manner similar to before, we see that the coordinate transformation fixes the solution for $\mathsf{R}(\tau, w)$, and that it matches the solution for $b(\tau, w)$ obtained directly from the 5-dimensional vacuum field equations (22b).

The solution for T is obtained without difficultly as before. For $k = \pm 1$, we get

$$\begin{aligned} \mathsf{T}(\tau, w) = {} & \frac{1}{k}\int_w \left\{ \frac{1}{\zeta(u)}\frac{d}{du}\chi(u) - \left[\frac{\chi(u)}{\zeta^2(u) - k}\right]\frac{d}{du}\zeta(u) \right\} du \, + \\ & \frac{1}{k}\left\{ \zeta(w)\tau - \frac{\mathcal{K}}{2\sqrt{k\mathcal{K}}}\ln\frac{1 + \mathsf{X}(\tau, w)}{1 - \mathsf{X}(\tau, w)} \right\}, \end{aligned} \tag{57a}$$

$$\mathsf{X}(\tau, w) \equiv \frac{k}{\sqrt{k\mathcal{K}}}\frac{[\zeta^2(w) - k]\tau + \chi(w)}{\zeta(w)}. \tag{57b}$$

For $k = 0$, we obtain

$$\begin{aligned} \mathsf{T}(\tau, w) = {} & \frac{1}{\mathcal{K}}\int_w \left\{ \frac{\chi^2(u)}{\zeta^3(u)}\frac{d}{du}\chi(u) - \frac{\chi(u)[\chi^2(u) - \mathcal{K}]}{\zeta^4(u)}\frac{d}{du}\zeta(u) \right\} du \, + \\ & \frac{1}{\mathcal{K}}\left\{ \frac{1}{3}\zeta^3(w)\tau^3 + \zeta(w)\chi(w)\tau^2 + \left[\frac{\chi^2(w) - \mathcal{K}}{\zeta(w)}\right]\tau \right\}. \end{aligned} \tag{58}$$

These transformations (equations 56–58) are extremely similar to the ones derived in the previous section. Just as before, there are special issues with the $k\mathcal{K} < 0$ case that can be dealt with using the identity (51); but we defer such a discussion as it does not add much to what we have established.

In conclusion, we have succeeded in finding a coordinate transformation from the TBH to FSW metrics. Since we have already found a transformation from TBH to LMW, this allows us to also conclude that a coordinate transformation between the FSW and LMW metrics exists as well.

C. Comment: the generalized Birkhoff theorem

Before moving on, we would like to make a comment about how the equivalence of the LMW, FSW, and TBH metrics relates to the issue of a generalized version of the Birkhoff theorem. Both the LMW and FSW *ansatzs* are of the general form:

$$ds^2 = A^2(t,\ell)\,dt^2 - B^2(t,\ell)\,d\sigma^2_{(k,3)} - C^2(t,\ell)\,d\ell^2. \tag{59}$$

To this line element, we can apply the coordinate transformation

$$R = B(t,\ell) \tag{60}$$

to obtain

$$ds^2 = P^2(t,R)\,dt^2 - R^2\,d\sigma^2_{(k,3)} - 2N(t,R)\,dt\,dR - Q^2(t,R)\,dR^2. \tag{61}$$

Here, P, Q and N are related to the original metric functions A, B, and C, but their precise form is irrelevant. Then, we apply the diffeomorphism

$$dt = M(T,R)\,dT + \frac{N(t,R)}{P^2(t,R)}\,dR, \tag{62}$$

where $M(T,R)$ is an integrating factor that should satisfy

$$\frac{1}{M}\frac{\partial M}{\partial R} = \frac{\partial}{\partial t}\frac{N}{P^2}, \tag{63}$$

in order to ensure that dt is a perfect differential. In these coordinates, the line element is

$$ds^2 = f(T,R)\,dT^2 - g(T,R)\,dR^2 - R^2\,d\sigma^2_{(k,3)}. \tag{64}$$

Again, f and g are determined by the original metric functions and the integrating factor. This structure is strongly reminiscent of the general spherically-symmetric metric from 4-dimensional relativity. The only difference is that the line element on a unit 2-sphere $d\Omega^2$ has been replaced by $d\sigma^2_{(k,3)}$. In the 4-dimensional case, Birkhoff's theorem tells us that the only solution to the vacuum field equations with the general spherically-symmetric line element is the Schwarzschild metric. The theorem has been extended to the multi-dimensional case by Bronnikov & Melnikov[17], who showed that the 5-dimensional vacuum solution with line element (64) is unique and given by the TBH metric. So, in retrospect it is perhaps apparent that the LMW, FSW, and TBH solutions are equivalent — any 5-dimensional vacuum solution that can be cast in the form of (59) must be isometric to the TBH metric. We

conclude by noting that this type of argument extends to the case of 5-dimensional Einstein spaces as well, because another variation of Birkhoff's theorem derived by Bronnikov & Melnikov is applicable. That is, if there is a cosmological constant in the bulk — as in the popular Randall & Sundrum braneworld models — a metric solution of the form (59) will be equivalent to a deSitter or anti-deSitter TBH manifold. For example, the "wave-like" solutions sourced by a cosmological constant found by Ponce de Leon[18] should be isometric to 5-dimensional Schwarzschild-AdS black holes.

V. PENROSE-CARTER DIAGRAMS OF FLRW MODELS EMBEDDED IN THE LIU-MASHHOON-WESSON METRIC

We have now established that the LMW, FSW, and TBH solutions of the vacuum field equations are mutually isometric. This means that they each correspond to coordinate patches on the same 5-dimensional manifold. Now, it is well-known that the familiar Schwarzschild solution in four dimensions only covers a portion of what is known as the extended Schwarzschild manifold[19]. It stands to reason that if there is a Killing horizon in the TBH metric, then the (T, R) coordinates will also only cover part of some extended manifold M. This raises the question: what portion of the extended manifold M is covered by the (t, ℓ) or (τ, w) coordinates? This is interesting because it is directly related to the issue of what portion of M is spanned by the universes embedded on the Σ_ℓ and Σ_w hypersurfaces.

We do not propose to answer these questions for all possible situations because there are a wide variety of choices of free parameters. We will instead concentrate on one particular problem: namely, the manner in which the Liu-Mashhoon-Wesson coordinates cover the extended manifold M when $k = +1$, $\mathcal{K} > 0$, and for specific choices of μ and ν. The restriction to spherical S_3 submanifolds means that the maximal extension of the (T, R) coordinate patch proceeds analogously to the 4-dimensional Kruskal construction. The calculation can be straightforwardly generalized to the Fukui-Seahra-Wesson coordinates if desired.

We first need to find the 5-dimensional generalization of Kruskal-Szekeres coordinates for the $k = +1$ TBH metric. (See ref. 20 for background information about the 4-dimensional formalism.) Let us apply the following transformations to the metric (33):

$$R_* = R + \frac{1}{2} m \ln \left| \frac{R - m}{R + m} \right|, \quad u = T - R_*, \quad v = T + R_*, \tag{65}$$

where we have defined $\mathcal{K} \equiv m^2$ such that the event horizon is at $R = m$. We then obtain

$$ds^2_{\text{BH}} = \operatorname{sgn} h(R) \, \frac{(R^2 + m^2) e^{-2R/m}}{R^2} e^{-u/m} e^{v/m} \, du \, dv - R^2 \, d\Omega_3^2. \tag{66}$$

Here, we have changed the "TBH" label to "BH" to stress that we are dealing with an ordinary black hole with spherical symmetry. This metric is singularity-free at $R = m$. The next transformation is given by

$$\tilde{U} = \mp \operatorname{sgn} h(R) \, e^{-u/m}, \quad \tilde{V} = \pm e^{v/m}, \tag{67}$$

which puts the metric in the form

$$ds^2_{\text{BH}} = m^2 \left(1 + \frac{m^2}{R^2} \right) e^{-2R/m} \, d\tilde{U} d\tilde{V} - R^2 \, d\Omega_3^2. \tag{68}$$

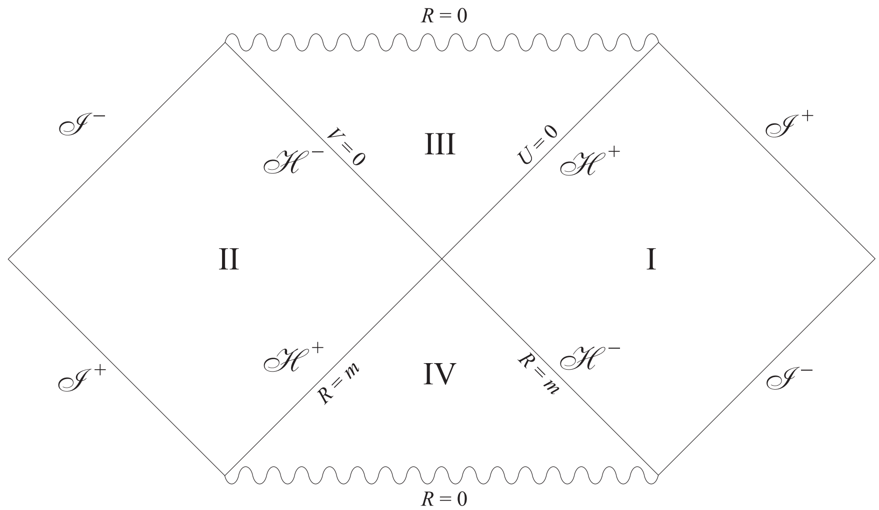

FIG. 1: Penrose-Carter diagram of a 5-dimensional black hole manifold

This is very similar to the 4-dimensional Kruskal-Szekeres coordinate patch on the Schwarzschild manifold. The aggregate coordinate transformation from (T, R) to $(\tilde{U}, \tilde{V})$ is given by

$$\tilde{U} = \mp \operatorname{sgn} h(R)\, e^{-T/m} e^{R/m} \sqrt{\left|\frac{R-m}{R+m}\right|}, \tag{69a}$$

$$\tilde{V} = \pm e^{T/m} e^{R/m} \sqrt{\left|\frac{R-m}{R+m}\right|}. \tag{69b}$$

From these, it is easy to see that the horizon corresponds to $\tilde{U}\tilde{V} = 0$. Now, what are we to make of the sign ambiguity in these coordinate transformations? Recall that in four dimensions, the extended Schwarzschild manifold involves two copies of the ordinary Schwarzschild spacetime interior and exterior to the horizon. It is clear that something analogous is happening here: the mapping $(T, R) \to (\tilde{U}, \tilde{V})$ is double-valued because the original (T, R) coordinates can correspond to one of two different parts of the extended manifold. This is best illustrated with a Penrose-Carter diagram, which is given in Figure 1. As is the usual practice, to obtain such a diagram we "compactify" the $(\tilde{U}, \tilde{V})$ coordinates by introducing

$$U = \frac{2}{\pi} \arctan \tilde{U}, \quad V = \frac{2}{\pi} \arctan \tilde{V}. \tag{70}$$

Figure 1 has all of the usual properties: null geodesics travel on 45° lines, the horizons appear at $U = 0$ or $V = 0$, the singularities show up as horizontal features at the top and bottom, and each point in the two-dimensional plot represents a 3-sphere. Also, in quadrant I the T-coordinate increases from bottom to top, while the reverse is true in quadrant II. We see that the top sign in the coordinate transformation (69) maps (T, R) into regions I or III of the extended manifold where $V > 0$, while the lower sign defines a mapping into II or IV where $V < 0$.

Having obtained the transformation to Kruskal-Szekeres coordinates, we can now plot the trajectory of the Σ_ℓ hypersurfaces through the extended manifold by using (45) and (48)

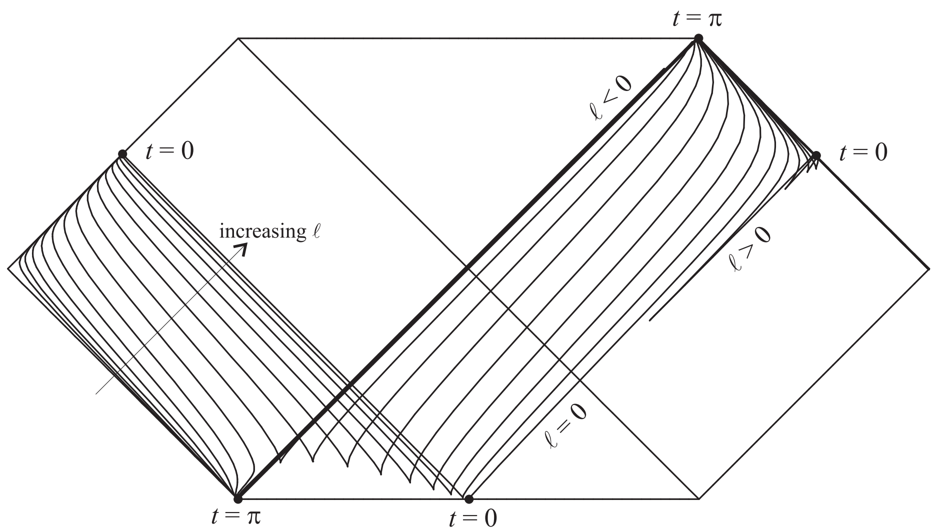

FIG. 2a: Σ_ℓ hypersurfaces of the LMW metric for the special choices (71). Each point in the Penrose-Carter diagram represents a 3-sphere. We restrict $t \in (0, \pi)$. The corresponding values of ℓ range from ~ -2.2 to 0.3 in equal logarithmic intervals. Note that even though the two points marked $t = \pi$ appear to be on the $U = 0$ line, they are actually located on $\mathcal{I}^+$ in region II and $R = 0$ in region III. This can be explicitly confirmed by greatly enlarging the scale of the plot.

in (69) to find $U(t, \ell)$ and $V(t, \ell)$. But there is one wrinkle: we need to flip the sign of the $(T, R) \to (U, V)$ transformation whenever the path crosses the $V = 0$ line, which is not hard to accomplish numerically. In Figure 2, we present Penrose-Carter embedding diagrams of Σ_ℓ and Σ_t hypersurfaces associated with the LMW metric for the following choices of parameters and free functions:

$$m = \tfrac{1}{2}, \quad \mu(t) = \cot t, \quad \nu(t) = \tfrac{\sqrt{3}}{2}. \tag{71}$$

This gives

$$a(t, \ell) = \sqrt{\left(\ell \csc t + \tfrac{\sqrt{3}}{2} \sin t\right)^2 + \tfrac{1}{4} \sin^2 t}. \tag{72}$$

Our choices imply that it is sensible to restrict $t \in (0, \pi)$. For $\ell \neq 0$, the cosmologies embedded on Σ_ℓ do *not* undergo a big bang or big crunch and $a \to \infty$ as $t \to 0$ or π. The $\ell = 0$ cosmology simply has $a(t, 0) = \sin t$. That is, we have a re-collapsing model. The induced metric for that hypersurface is

$$ds^2_{(\Sigma_0)} = \sin^2 t \, (dt^2 - d\Omega_3^2), \tag{73}$$

that of a closed radiation-dominated universe.

In Figure 2a we show the Σ_ℓ hypersurfaces of this model in a Penrose-Carter diagram. In this plot it is easy to visually determine where each trajectory begins when $t = 0$, but because of the scale it is difficult to note precisely where they end up at when $t = \pi$. By careful analysis of the numeric results, we have determined the following facts: The $\ell = 0$ trajectory emanates from the middle of the singularity in the white hole region IV at $t = 0$ and terminates on the future singularity in region III at $t = \pi$. The surfaces with $\ell > 0$

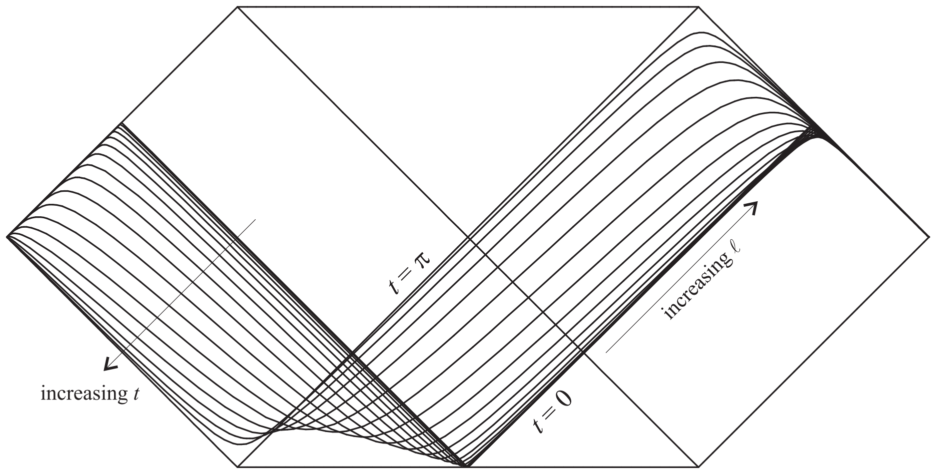

FIG. 2b: Isochrones of the LMW metric for the special choices (71). We restrict $\ell \in (-5, 5)$. The corresponding values of t range from 0 to $\sim \pi/2$ in equal logarithmic intervals. A portion of the $t = \pi$ surface is also shown, which appears to be coincident with $\mathcal{H}^+$. However, in reality it is only parallel to $U = 0$, but the finite separation between the two surfaces is impossible to discern without greatly enlarging the scale of the plot.

begin at $\mathcal{I}^+$ in I and terminate on $\mathcal{I}^-$ in the same region. The models with $\ell < 0$ all begin on $\mathcal{I}^-$ and terminate on $\mathcal{I}^+$ in region II. We mention in passing that this plot bears some qualitative resemblance to the figures of Mukohyama et al.[21], who showed the equivalence of a known solution of the 5-dimensional field equations with a cosmological constant and the topological Schwarzschild-AdS black hole in the context of braneworld scenarios; but many details are significantly different.

One of the most striking features of this plot is the cusps present in the majority of the Σ_ℓ curves. These sharp corners suggest some sort of singularity in the embedding at their location. We can search for the singularity by examining scalars formed from the extrinsic curvature of the Σ_ℓ 4-surfaces. Let us consider

$$h^{\alpha\beta} K_{\alpha\beta} = \frac{a_{,t\ell}}{a_{,t}} + 3\frac{a_{,\ell}}{a}. \tag{74}$$

One can confirm directly that this diverges whenever $a_{,t} = 0$ and $a_{,t\ell} \neq 0$. At such positions, we find sharp corners in the Σ_ℓ hypersurfaces. This makes it clear that if we wanted to use the LMW coordinates as a patch on the extended 5-dimensional black hole manifold, we would have to restrict t to lie in an interval bounded by times defined by the turning points of a. This is in total concurrence with the analysis of singularities in the intrinsic 4-geometry performed in Section II A — the cusps correspond to singularities in the induced metric on Σ_ℓ. Actually we have confirmed that the curves with cusps generally have two curvature anomalies, but those additional features tend to get compressed into a region too small to resolve in Figure 2a. What is also interesting about these plots is how the LMW metric occupies a fair bit of territory in M (some of the Σ_ℓ hypersurfaces span regions I, II and IV). Like the Kruskal-Szekeres coordinates, the LMW patch is regular across the horizon(s).

The exact portion of the extended manifold spanned by our model is a little clearer in Figure 2b. In this plot, we show the Σ_t spacelike hypersurfaces — or isochrones — of the LMW metric. These stretch from spacelike infinity in region II to a point on $\mathcal{I}^+$ in region

I. The LMW time t is seen to run from bottom to top in I and *vice versa* in II. We also see clearly that there is a portion of the white hole region IV that is not covered by the LMW metric with $t \in (0, \pi)$. The $t = \pi$ line appears to coincide with $U = 0$, but is in actuality displaced slightly to the left. Notice that the area bounded by the $t = \pi/2$ and $t = \pi$ curves is relatively small, from which it follows that the portions of the Σ_ℓ surfaces with $\pi/2 \lesssim t \lesssim \pi$ tend to occupy an extremely compressed portion of the embedding diagram.

In summary, we have presented embedding diagrams for the Σ_ℓ and Σ_t hypersurfaces associated with the LMW metric in the Penrose-Carter graphical representation of the extended 5-dimensional black hole manifold. This partially answers the question of which portion of M is occupied by the LMW metric. However, the calculation was for specific choices of μ, ν, and $\mathcal{K}$. We have no doubt that more general conclusions are attainable, but that is a subject for a different venue.

VI. SUMMARY AND DISCUSSION

In this paper, we introduced two solutions of the 5-dimensional vacuum field equations, the Liu-Mashhoon-Wesson and Fukui-Seahra-Wesson metrics, in Sections II A and II B respectively. We showed how both of these embed certain types of FLRW models and studied the coordinate invariant properties of the associated 5-manifolds. We found that both solutions had line-like curvature singularities and Killing horizons, and that their Kretschmann scalars were virtually identical. These coincidences prompted us to suspect that the LMW and FSW metrics are actually equivalent, and that they are also isometric to the 5-dimensional topological black hole metric introduced in Section III. This was confirmed explicitly in Section IV, where transformations from Schwarzschild-like to LMW and FSW coordinates were derived. The strategy employed in that section was to transform the TBH line element into the form of the LMW and FSW metric *ansatzs*, which resulted in two sets of solvable PDEs. Therefore, those calculations comprise independent derivations of the LMW and FSW metrics. In Section IV C, we showed how the relationship between the LMW, FSW and TBH metrics was a consequence of a generalized version of Birkhoff's theorem. Finally, in Section V we performed a Kruskal extension of the 5-dimensional black hole manifold and plotted the Σ_ℓ and Σ_t hypersurfaces of the LMW metric in a Penrose-Carter diagram for certain choices of μ, ν, and $\mathcal{K}$.

Obviously, our main result is that the LMW and FSW metrics are non-trivial coordinate patches on 5-dimensional black hole manifolds. We saw explicitly that the LMW coordinates could cover multiple quadrants of the maximally-symmetric manifold, and that they were regular across the event horizon. This puts them in the same category as the Eddington-Finkelstein (EF) or Painlevé-Gullstrand (PG) coordinates associated with 4-dimensional Schwarzschild black holes[22], which are also horizon piercing patches that do not involve implicit functions, such as $R = R(U, V)$ in the Kruskal-Szekeres covering. The LMW coordinates differ from the EF or PG patches in that they are 5-dimensional and orthogonal. All of these features make them an attractive tool for the study of black hole physics in 5 dimensions. In particular, they provide "rest-frame" coordinates for embedded 4-dimensional universes. That is, in both the LMW and FSW coordinates, universes are defined simply as 4-surfaces comoving in ℓ or w. And unlike standard Schwarzschild-like coordinates, the LMW or FSW 5-metrics are regular as the universe crosses the black hole horizon(s). Such coordinates may have some utility in the study of quantized braneworld models, where the bad behaviour of coordinates across horizons apparently results in a complicated canonical

phase-space description of the brane's dynamics[23].

Finally, we discuss the temptation to generalize these coordinates to other types of black holes and different dimensions. One could easily imagine repeating the manipulations of Section IV for different choices of $h(R)$, which could be selected to correspond to any spherically-symmetric black hole in any dimension. However, a difficulty arises when one tries to integrate equations like

$$\mathcal{R}_{,\ell} = \pm\sqrt{h(\mathcal{R}) + \mu^2(t)}. \tag{75}$$

to obtain $\mathcal{R} = \mathcal{R}(t, \ell)$ explicitly. It turns out that this in not necessarily easy to do if $h(\mathcal{R}) \neq k - \mathcal{K}/\mathcal{R}^2$. For example, if h corresponds to an N-dimensional topological black hole (i.e., $h(\mathcal{R}) = k - \mathcal{K}/\mathcal{R}^{N-3}$) we obtain complicated implicit definitions of $\mathcal{R}$ involving generalized hypergeometric functions. For even $N = 4$, it is unclear how to invert such an equation to find $\mathcal{R} = \mathcal{R}(t, \ell)$ explicitly. So it seems that the 5-dimensional case is somewhat special. However, we do not preclude the possibility that there are other special cases out there, that our procedure could be improved upon, or that one could find suitable coordinates by direct assault on the N-dimensional field equations. Such issues are best addressed by future work.

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